Simple Co-ordinate geometry problems

1. Find the equation of straight line passing through the point $P(5,2)$ with equal intercepts.

1. **Method 1**

Let the equation of straight line be $\frac{x}{a}+\frac{y}{b}=1, \quad a, b \neq 0$

(a) If $a=b \neq 0$ and the straight line passes through $P(5,2)$,

$$\frac{5}{a}+\frac{2}{a}=1 \quad \text{and} \quad a=7$$

The straight line is $x+y=7$.

(b) If $a=b=0$, the straight line passes through the origin and the equation is $2x-5y=0$.

Method 2

Let the equation of straight line passing through the point $P(5,2)$ be

$$y-2=m(x-5)$$

Since x-intercept = y-intercept

$$-\frac{2}{m}+5=2-5m \quad \text{, where } m \neq 0$$

$$5m^2-3m-2=0$$

$$\therefore m=-\frac{2}{5} \text{ or } m=1$$

The equations are $y-2=-\frac{2}{5}(x-5)$ or $y-2=1(x-5)$

$$2x-5y=0 \quad \text{or} \quad 2x-5y=0 \; .$$

2. Consider a circle with center (4,5) and with radius of 8

If the tangent lines of the circle has slope $\frac{1}{3}$. Find :

(a) the points of contact of these tangent lines and the circle,

(b) the equation of the tangent lines.

2. **(a)**

Method 1

The circle is C: $(x-4)^2+(y-5)^2=64$

Since the tangent slope is $\frac{1}{3}$, the slope of the line perpendicular to the tangent is -3.

Let this diameter line passes through the centre $(4,5)$ be L. Then

$$L:\; y-5=-3(x-4)$$

Solving, $\begin{cases} \text{C: } (x-4)^2 + (y-5)^2 = 64 & \text{....(1)} \\ \text{L: } y-5 = -3(x-4) & \text{....(2)} \end{cases}$

(2) ↓ (1), $(x-4)^2 + 9(x-4)^2 = 64$

$10(x-4)^2 = 64$

$x - 4 = \frac{\pm 8}{\sqrt{10}} = \frac{\pm 4\sqrt{10}}{5}$

$x = \frac{20 \pm 4\sqrt{10}}{5}$ and $y = \frac{25 \pm 12\sqrt{10}}{5}$

The required points are $\left(\frac{20+4\sqrt{10}}{5}, \frac{25+12\sqrt{10}}{5}\right)$ or $\left(\frac{20-4\sqrt{10}}{5}, \frac{25-12\sqrt{10}}{5}\right)$

Method 2

The circle is C: $(x-4)^2 + (y-5)^2 = 64$

Differentiate with respect to x,

$2(x-4) + 2(y-5)\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x-4}{y-5}$

Since the tangent slope is $\frac{1}{3}$, $-\frac{x-4}{y-5} = \frac{1}{3}$

We get a straight line L: $y = -3x + 17$

(This line is **not** the tangent line, but the diameter line passing (4,5) and the point of contact. In fact we just get a line which is good for the given.)

Substitute in C, we have $(x-4)^2 + (-3x + 17 - 5)^2 = 64$

$10x^2 - 80x + 96 = 0$ or $5x^2 - 40x - 48 = 0$

$x = \frac{20 \pm 4\sqrt{10}}{5}$ and $y = \frac{25 \pm 12\sqrt{10}}{5}$

The required points are $\left(\frac{20+4\sqrt{10}}{5}, \frac{25+12\sqrt{10}}{5}\right)$ or $\left(\frac{20-4\sqrt{10}}{5}, \frac{25-12\sqrt{10}}{5}\right)$

Method 3

The circle in parametric form is C: $\begin{cases} x = 4 + 8\cos\theta \\ y = 5 + 8\sin\theta \end{cases}$, $0 \le \theta < 360°$

$$\begin{cases} \frac{dx}{dt} = -8\sin\theta \\ \frac{dy}{dt} = 8\cos\theta \end{cases} \Rightarrow \frac{dy}{dx} = \frac{8\cos\theta}{-8\sin\theta} = \cot\theta$$

Since the tangent slope is $\frac{1}{3}$, $\cot\theta = \frac{1}{3} \Rightarrow \tan\theta = 3 \Rightarrow \begin{cases} \cos\theta = \pm\frac{1}{\sqrt{10}} = \pm\frac{\sqrt{10}}{10} \\ \sin\theta = \pm\frac{3}{\sqrt{10}} = \pm\frac{3\sqrt{10}}{10} \end{cases}$

$$\therefore \begin{cases} x = 4 + 8\cos\theta = 4 + 8\left(\pm\frac{\sqrt{10}}{10}\right) \\ y = 5 + 8\sin\theta = 5 + 8\left(\pm\frac{3\sqrt{10}}{10}\right) \end{cases}$$

The required points are $\left(\frac{20+4\sqrt{10}}{5}, \frac{25+12\sqrt{10}}{5}\right)$ or $\left(\frac{20-4\sqrt{10}}{5}, \frac{25-12\sqrt{10}}{5}\right)$.

Method 4

The circle in parametric form is C: $\begin{cases} x = 4 + 8\cos\theta \\ y = 5 + 8\sin\theta \end{cases}, \ 0 \le \theta < 360°$

Let the pencil of straight lines with slope $\frac{1}{3}$ be L: $y = \frac{1}{3}x + c$

Hence, we have: $5 + 8\sin\theta = \frac{1}{3}(4 + 8\cos\theta) + c$

$8\cos\theta - 24\sin\theta = 11 - 3c$

Writing in subsidiary angles, $\sqrt{8^2 + 24^2}\cos(\theta - \alpha) = 11 - 3c$ where $\tan\alpha = \frac{24}{8} = 3$

Since the straight line touches the circle at only one point, we have $\cos(\theta - \alpha) = 1 \text{ or } -1$

Therefore, $\pm\sqrt{8^2 + 24^2} = 11 - 3c$

$$c = \frac{11+8\sqrt{10}}{3}$$

(b) Method 1

Use point-slope form, in **(a)** we get slope $= \frac{1}{3}$, Points of contact $= \left(\frac{20\pm4\sqrt{10}}{5}, \frac{25\pm12\sqrt{10}}{5}\right)$

Hence equations of tangent are:

$$y - \frac{25\pm12\sqrt{10}}{5} = \frac{1}{3}\left(x - \frac{20\pm4\sqrt{10}}{5}\right)$$

$$y = \frac{1}{3}x + \frac{11\pm8\sqrt{10}}{3}$$

If there is no need to find the points of contact, the following methods are also of interest.

Method 2

Let the pencil of straight lines with slope $\frac{1}{3}$ be L: $y = \frac{1}{3}x + c \quad \ldots.(1)$

It touches the circle C: $(x-4)^2 + (y-5)^2 = 64 \quad \ldots.(2)$

(1) ↓ (2), $(x-4)^2 + \left[\left(\frac{1}{3}x + c\right) - 5\right]^2 = 64$

$$\frac{10x^2}{9} + \left(\frac{2c}{3} - \frac{34}{3}\right)x + (c^2 - 10c - 23) = 0$$

Since L touches C at one point, $\Delta = \left(\frac{2c}{3} - \frac{34}{3}\right)^2 - 4 \times \frac{10}{9} \times (c^2 - 10c - 23) = 0$

$$-4\,c^2+\frac{88\,c}{3}+\frac{692}{3}=0$$

$$12\,c^2-88c-692=0$$

$$c=\frac{11+8\sqrt{10}}{3} \text{ or } c=\frac{11-8\sqrt{10}}{3}$$

Hence the equations of tangent are: $\quad y=\frac{1}{3}x+\frac{11\pm 8\sqrt{10}}{3}$

Method 3

The circle in parametric form is C: $\begin{cases} x=4+8\cos\theta \\ y=5+8\sin\theta \end{cases}$, $0\le\theta<360°$

Let the pencil of straight lines with slope $\frac{1}{3}$ be L: $y=\frac{1}{3}x+c$

Hence, we have: $\quad 5+8\sin\theta=\frac{1}{3}(4+8\cos\theta)+c$

$8\cos\theta-24\sin\theta=11-3c$

Writing in subsidiary angles, $\sqrt{8^2+24^2}\cos(\theta-\alpha)=11-3c$ where $\tan\alpha=\frac{24}{8}=3$

For each c we may have no solution, one solution or two solutions for the straight line.
Since the straight line touches the circle at only one point, we have $\cos(\theta-\alpha)=1$ or -1.
(For $\cos(\theta-\alpha)=0$, the straight line cuts the circle at two points! You get a diameter equation which is parallel to the tangent lines we are interested. Investigate yourselves.))
Therefore, $\pm\sqrt{8^2+24^2}=11-3c$

$$c=\frac{11+8\sqrt{10}}{3}$$

Hence the equations of tangent are: $\quad y=\frac{1}{3}x+\frac{11\pm 8\sqrt{10}}{3}$

3. Find the value(s) of m such that $(2m^3+m^2-m)x^2+(m^3-m^2+2m)y^2-8m+18=0$ represents the equation of a circle.

3. $Ax^2+By^2+C=0$ represents a circle if and only if $A=B\neq 0$ and the radius > 0.
Now, $\quad 2m^3+m^2-m=m^3-m^2+2m$

$m^3+2m^2-3m=0$

$m\,(m-1)\,(m+3)=0$

$m=0$ or $m=1$ or $m=-3$

(a) When $m=0$, $2m^3+m^2-m=m^3-m^2+2m=0$, no equation existed and $m=0$ is rejected.

(b) When $m = 1$, The circle is $2x^2 + 2y^2 + 10 = 0$ or $x^2 + y^2 + 5 = 0$
The radius $=\sqrt{-5}$ which is imaginary and $m = 1$ is rejected.

(c) When $m = -3$,
The circle is $-42x^2 - 42y^2 + 42 = 0$ or $x^2 + y^2 - 1 = 0$
The radius is 1 . This satisfies all conditions of the given.

4. Find the equation of a circle C_1 passing through the intersection points of

$$\begin{cases} L:\ x - 3y + 4 = 0 \\ C: x^2 + y^2 + 2x - 6y + 2 = 0 \end{cases}$$

and with the smallest area.

4. **Method 1**

The intersection point of $\begin{cases} L:\ x - 3y + 4 = 0 \\ C: x^2 + y^2 + 2x - 6y + 2 = 0 \end{cases}$

are $\left(\frac{-3\sqrt{11}-2}{5}, \frac{6-\sqrt{11}}{5}\right)$ or $\left(\frac{3\sqrt{11}-2}{5}, \frac{\sqrt{11}+6}{5}\right)$ (working steps omitted)

The smallest circle that can be formed should use these two points as diameter.
Using the diameter form, the required circle is:

$$C_1: \frac{y-\frac{6-\sqrt{11}}{5}}{x-\frac{-3\sqrt{11}-2}{5}} \times \frac{y-\frac{\sqrt{11}+6}{5}}{x-\frac{3\sqrt{11}-2}{5}} = -1$$

$$C_1: \left(y - \frac{6-\sqrt{11}}{5}\right)\left(y - \frac{\sqrt{11}+6}{5}\right) + \left(x - \frac{-3\sqrt{11}-2}{5}\right)\left(x - \frac{3\sqrt{11}-2}{5}\right) = 0$$

$$C_1: x^2 + y^2 + \frac{4x}{5} - \frac{12y}{5} - \frac{14}{5} = 0$$

Method 2

Let the system of circles passing through L and C be

$$C_1:\ x^2 + y^2 + 2x - 6y + 2 + \lambda(x - 3y + 4) = 0$$
$$C_1:\ x^2 + y^2 + (2 + \lambda)x - (6 + 3\lambda)y + (2 + 4\lambda) = 0$$

(**Method 2A)** If this circle has the smallest area, the radius r is also smallest.

$$r^2 = \left(-\frac{2+\lambda}{2}\right)^2 + \left(\frac{6+3\lambda}{2}\right)^2 - (2 + 4\lambda) = \frac{5\lambda^2+12\lambda+16}{2} = \frac{5}{2}\left(\lambda^2 + \frac{12}{5}\lambda + \frac{16}{5}\right)$$
$$= \frac{5}{2}\left[\left(\lambda + \frac{6}{5}\right)^2 - \frac{36}{25} + \frac{16}{2}\right] = \frac{5}{2}\left[\left(\lambda + \frac{6}{5}\right)^2 + \frac{164}{25}\right]$$

Hence r is smallest when $\lambda = -\frac{6}{5}$.

(**Method 2B)** If this circle has the smallest area, the radius r is also smallest.

Hence the centre of $C_1 = \left(-\frac{2+\lambda}{2}, \frac{6+3\lambda}{2}\right)$ must be on the line L.

We therefore have $-\frac{2+\lambda}{2} - 3\left(\frac{6+3\lambda}{2}\right) + 4 = 0$

Hence r is smallest when $\lambda = -\frac{6}{5}$.

The required circle is:

$$C_1:\ x^2 + y^2 + \left(2 - \frac{6}{5}\right)x - \left(6 + 3\left(-\frac{6}{5}\right)\right)y + \left(2 + 4\left(-\frac{6}{5}\right)\right) = 0$$

$$C_1: x^2 + y^2 + \frac{4x}{5} - \frac{12y}{5} - \frac{14}{5} = 0$$

Method 3

The centre, G, of C = $(-1,3)$

Let L_1 be the line perpendicular to L and passing through G.

Gradient of L = $\frac{1}{3}$ and Gradient of $L_1 = -3$.

Hence, $L_1: y - 3 = -3(x + 1)$ or $y = -3x$

Solving L and L_1, we get $A\left(-\frac{2}{5}, \frac{6}{5}\right)$. This is the centre of the required circle.

Let the system of circles passing through L and C be

$$C_1:\ x^2 + y^2 + 2x - 6y + 2 + \lambda(x - 3y + 4) = 0$$
$$C_1:\ x^2 + y^2 + (2 + \lambda)x - (6 + 3\lambda)y + (2 + 4\lambda) = 0$$

The centre is also $A\left(-\frac{2+\lambda}{2}, \frac{6+3\lambda}{2}\right) = \left(-\frac{2}{5}, \frac{6}{5}\right)$

$-\frac{2+\lambda}{2} = -\frac{2}{5}$ and $\lambda = -\frac{6}{5}$. The required circle is:

$$C_1:\ x^2 + y^2 + \left(2 - \frac{6}{5}\right)x - \left(6 + 3\left(-\frac{6}{5}\right)\right)y + \left(2 + 4\left(-\frac{6}{5}\right)\right) = 0$$

$$C_1: x^2 + y^2 + \frac{4x}{5} - \frac{12y}{5} - \frac{14}{5} = 0$$